Retake Probability Theory (WIKR-06)

12 July 2018, 09.00 – 12.00

- Every exercise needs to be handed in on separate sheets, which will be collected in separate piles.
- Write your name and student number **on every sheet**.
- It is absolutely not allowed to use calculators, phones, the book, notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question this is not needed.).
- **NOTA BENE:** using separate sheets for the different exercises and writing your name and student number on all sheets is worth 10 out of the 100 points.

Exercise 1, (a:6, b:6, c:6, d:6, e:6 pts).
The random variables X, Y, Z have a joint pmf given by :

$$p_{X,Y,Z}(x,y,z) = \begin{cases} 1-7a & \text{if } (x,y,z) = (0,0,0); \\ a & \text{if } (x,y,z) \in \{0,1\}^3 \setminus \{(0,0,0)\}, \\ 0 & \text{otherwise.} \end{cases}$$

(where of course $0 < a < \frac{1}{7}$.)

(a) Determine the (marginal) pmfs of X, Y and Z;

(b) Determine $\mathbb{E}X$ and $\mathbb{E}(X|Y=i, Z=j)$ for $i, j \in \{0,1\}$, and $\mathbb{E}(X|Y=i)$ for $i \in \{0,1\}$;

(c) Determine $\text{Cov}(X, Y)$;

(d) Determine for which values of a (if any) X, Y are independent.
(Nota bene: make sure to supply proofs of dependence/independence here. Just claiming "X, Y are (in)dependent for such and such values of a" will not suffice)

(e) Determine the cdf of $X + Y + Z$.

Exercise 2 (a:5, b:5, c:5, d:5, e:5, f:5 pts)
Recall that the gamma function is defined by

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t}\, dt,$$

(for $\alpha > 0$) and that the Gamma(α, β)-distribution has pdf

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(where $\alpha, \beta > 0$).

In this question we will explore the relation between the gamma distribution and the normal distribution.

(a) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(Hint: use the substitution $t = x^2/2$. You may use without proof that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1$. Make sure to justify all steps of your argument and calculations.)

(b) Now let X be standard normal. Show that $X^2 \sim$ Gamma$(\frac{1}{2}, 2)$.
(Hint: you may want to use that, for $x \geq 0$: $F_{X^2}(x) = \mathbb{P}(X^2 \leq x) = F_X(\sqrt{x}) - F_X(-\sqrt{x})$.)

(c) For $\alpha_1, \alpha_2, \beta > 0$, let $X_1 \sim$ Gamma$(\alpha_1, \beta), X_2 \sim$ Gamma(α_2, β) be idependent and set $X := X_1 + X_2$. Show that

$$f_X(x) = \begin{cases} \frac{x^{\alpha_1+\alpha_2-1} e^{-x/\beta}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \cdot \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\, dt & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: If you remember it, you may use without proof the convolution formula. Otherwise you could use the familiar two-dimensional transformation $X = X_1 + X_2, Y = X_1$ and then integrate over Y.)

(d) Explain why $\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\, dt = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$.
(Hint: Use 2c. With the correct reasoning, no complicated calculations are necessary.)

(e) Suppose that $X_1, \ldots, X_n$ are independent and standard normal. Show that $X_1^2 + \cdots + X_n^2 \sim$ Gamma$(n/2, 2)$.

The Gamma$(n/2, 2)$-distribution is also called the *chi-squared distribution with n degrees of freedom*, denoted $\chi^2(n)$, and plays an important role in statistics.

(f) What is another name for the $\chi^2(2)$-distribution?
(Hint: The correct answer suffices. Note that, while in principle correct, we do not mean the answer "Gamma$(1, 2)$".)

Exercise 3 (a:7, b:8, c:7, d:8 pts)
A dog and a cat are sleeping very close to each other.

A flea (not shown in the picture above) is initially on one of the two animals. Every second the flea decides to either stay put or it jumps to the other animal, with probability p, independently of its choices in the previous seconds. We will be interested in the position of the flea after n seconds have passed. Let us write

$$p_n := \mathbb{P}(\text{the flea is on the cat at time } n) \quad (n = 1, 2, \dots).$$

and

$$p_0 := \begin{cases} 1 & \text{if the flea starts on the cat,} \\ 0 & \text{if the flea starts on the dog.} \end{cases}$$

(We leave open which of the two scenarios we are actually in.)

(a) Give a concise, yet rigorous, justification of the recursive relation

$$p_{n+1} = (1-p)p_n + p(1-p_n) \quad (n = 0, 1, \dots).$$

(b) By solving this recursion, or otherwise, show that

$$p_n = \frac{1}{2}\left(1 - (1-2p)^n\right) + p_0 \cdot (1-2p)^n.$$

(c) Let $X_n \sim \mathrm{Bin}(n, p)$. Give a "closed form" (that is without the use of the summation sign $\sum$ or integral sign $\int$ etc.) expression for

$$\mathbb{P}(X_n \text{ is even}) = \sum_{\substack{0 \le k \le n, \\ k \text{ even}}} \binom{n}{k} p^k (1-p)^{n-k}.$$

And, in particular show that $\mathbb{P}(X_n \text{ is even}) \overset{n\to\infty}{\longrightarrow} \frac{1}{2}$.

(d) This particular flea actually prefers the blood of the cat over the blood of the dog. So it will be more likely to jump when it is on the dog compared to when it is on the cat. Let us thus consider the situation as above except that we have two probabilities $0 < p_{\text{cat}} < p_{\text{dog}} < 1$ such that if the flea is on the cat there is a probability p_{cat} that it jumps to the dog in a given second, and similarly if the flea is on the dog it jumps with probability p_{dog}. Defining p_n as before, give an exact expression for p_n and show that

$$p_n \overset{n\to\infty}{\longrightarrow} \frac{p_{\text{dog}}}{p_{\text{cat}} + p_{\text{dog}}},$$

no matter on which animal the flea starts.